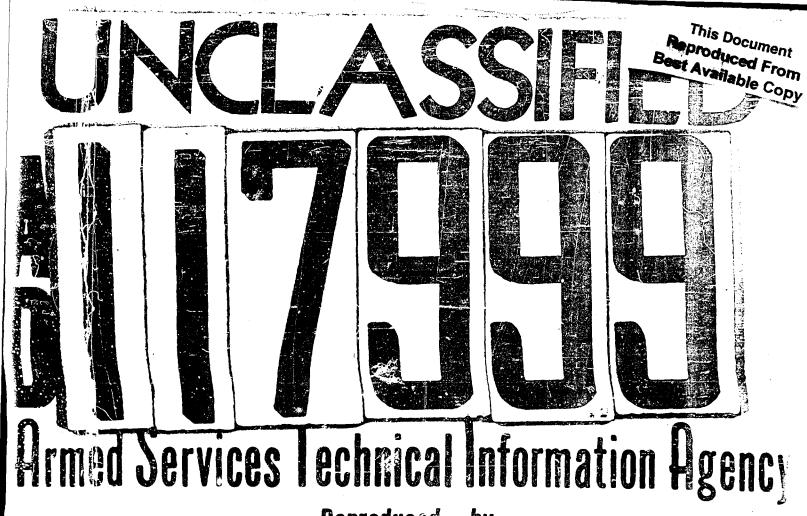THIS REPORT HAS BEEN DELIMITED
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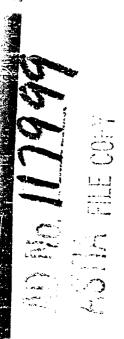
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A SIMPLE PROOF AND SOME EXTENSIONS
OF THE SAMPLING THEOREM

BY

EMANUEL PARZEN

TECHNICAL REPORT NO. 7

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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DECEMBER 22, 1956

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Abstract.

The sampling theorem states essentially that if the frequency spectrum, or Fourier transform, $g(\omega)$ of a time function $f(t)$ vanishes for ω outside some interval I , then $f(t)$ is completely determined by its values at certain discrete sampling points, whose density is proportional to the length of the interval I . This note gives a method of proof of the sampling theorem, both for the case where the interval I is centered at the origin and where it is not, which is somewhat simpler than the previously given proofs, and at the same time is more rigorous, and yields several useful generalizations to functions of several variables and random functions.

1.   Introduction.

The sampling theorem, as first stated by Shannon [1], asserts essentially that if the frequency spectrum, or Fourier transform, $g(\omega)$ of a time function $f(t)$ vanishes for ω outside some interval I , then $f(t)$ is completely determined by its values at certain discrete sampling points, whose density is proportional to the length of the interval. The usual proof of the sampling theorem is based on expanding the Fourier transform $g(\omega)$ in a Fourier series. However, it appears that the result can be obtained more simply by expanding the kernel $e^{i\omega t}$ in a Fourier series, as a function of ω in I , which amounts to proving the sampling theorem for the functions of t given by $e^{i\omega t}$, with ω held fixed at some point in I . In this note, we use this

method to prove the sampling theorem. We also show how the method may be used to give simple proofs of various extensions of the sampling theorem, among them those due to Goldman [2], Kohlenberg [3], and Woodward and Davies [4]. The theorems we prove are the following.

Theorem I: Suppose that the function $f(t)$, defined for all real values of t, may be represented as a Fourier integral, for some positive constant W,

$$(1.1) \qquad f(t) = \int_{-2\pi W}^{2\pi W} e^{i\omega t} g(\omega) \, d\omega$$

where the function $g(\omega)$ may contain δ function terms, but not at $\omega = \pm 2\pi W$. Then, for every t,

$$(1.2) \qquad f(t) = \sum_{m=-\infty}^{\infty} f(\tfrac{m}{2W}) \, s(t - \tfrac{m}{2W})$$

where

$$(1.3) \qquad s(t) = \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} e^{i\omega t} \, d\omega$$

$$(1.4) \qquad = \frac{\sin 2\pi W t}{2\pi W t}$$

Theorem II: Suppose that $f(t_1, \ldots, t_K)$, defined as a function of K variables, may be represented as a Fourier integral, for positive $W_1, \ldots, W_K$,

$$(1.5) \quad f(t_1, \ldots, t_K) = \int_{-2\pi W_1}^{2\pi W_1} \cdots \int_{-2\pi W_K}^{2\pi W_K} d\omega_1 \ldots d\omega_K e^{i(\omega_1 t_1 + \ldots + \omega_K t_K)} g(\omega_1, \ldots, \omega_K)$$

where $g(\omega_1, \ldots, \omega_K)$ may contain δ function terms, but not at the 2^K points of the form $(\pm 2\pi W_1, \ldots, \pm 2\pi W_K)$. Then

$$(1.6) \quad f(t_1, \ldots, t_K) = \sum_{m_1, \ldots, m_K = -\infty}^{\infty} f(\frac{m_1}{2W_1}, \ldots, \frac{m_K}{2W_K}) s(t_1 - \frac{m_K}{2W_K}) \ldots s(t_K - \frac{m_K}{2W_K})$$

Theorem III: Let $f(t)$ be a stationary random function (as defined by Doob [5]) whose covariance (or autocorrelation) function $R(\tau) = Ef(t) f(t+\tau)$ may be written,

$$(1.7) \qquad R(\tau) = \int_{-2\pi W}^{2\pi W} e^{i\omega\tau} G(\omega) \, d\omega$$

where $G(\omega)$ is the power spectrum of the function, which may contain δ function terms, but not at $\omega = \pm 2\pi W$. Then

$$(1.8) \qquad f(t) = \sum_{m=-\infty}^{\infty} f(\frac{m}{2W}) s(t - \frac{m}{2W})$$

where the infinite series in (18) converges stochastically (in the sense of convergence in mean square).

Theorem IV: Suppose that $f(t)$ may be represented as a Fourier integral, for positive W_0 and W,

(1.9) $\quad f(t) = \int_{2\pi W_o}^{2\pi(W_o + W)} e^{i\omega t} g(\omega) + \int_{-2\pi(W_o + W)}^{-2\pi W_o} e^{i\omega t} g(\omega) \, d\omega$

where the function $g(\omega)$ may possess δ function terms, but not at $\omega = \pm 2\pi W_o$ or $\omega = \pm 2\pi(W_o + W)$ or $\omega = \pm 2\pi(rW - W_o)$ where r is the integer such that

(1.10) $\qquad\qquad\qquad 2\dfrac{W_o}{W} \leq r < 2\dfrac{W_o}{W} + 1 \quad .$

Let

$$s(t) = \frac{1}{2\pi W} \int_{2\pi W_o}^{2\pi(rW + W_o)} d\omega[\cos \omega t + \sin \omega t \cot \pi rWk]$$

(1.11)

$$+ \frac{1}{2\pi W} \int_{2\pi(rW - W_o)}^{2\pi(W_o + W)} d\omega[\cos \omega t + \sin \omega t \cot \pi(r + 1)Wk]$$

where k is a constant satisfying the condition that rWk and $(r + 1)Wk$ are not equal to any of the integers $0, \pm 1, \ldots$ Then, for every t,

(1.12) $\quad f(t) = \displaystyle\sum_{m=-\infty}^{\infty} f(\tfrac{m}{W}) \, s(t - \tfrac{m}{W}) + f(\tfrac{m}{W} + k) \, s(\tfrac{m}{W} + k - t) \quad .$

If W_o/W is an integer, then with $k = 1/2W$, (1.12) reduces to (1.4).

It will be clear from what follows that Theorems II and III can be extended in the same way that Theorem I is extended by Theorem IV.

2. Proofs of Theorems I-III.

We start with the following basic fact from the theory of Fourier series. For any real number t, and ω such that $|\omega| < 2\pi W$,

$$(2.1) \qquad e^{i\omega t} = \sum_{m=-\infty}^{\infty} e^{i\omega \frac{m}{2W}} s(t - \frac{m}{2W}) \qquad .$$

The infinite series in (2.1) converges in general for $|\omega| < 2\pi W$, but not for $\omega = \pm 2\pi W$. However, the consecutive sums

$$S_M(\omega, t) = \sum_{|m| \le M} e^{i\omega \frac{m}{2W}} s(t - \frac{m}{2W})$$

are, for fixed t, uniformly bounded for $|\omega| \le 2\pi W$; that is, there is a constant K such that $|S_M(\omega,t)| \le K$ for $M = 0,1,2, \dots$ and $|\omega| \le 2\pi W$.

A not quite rigorous way of verifying (2.1) is as follows. The right hand side of (2.1) may be written

$$(2.2) \qquad \frac{1}{4\pi W} \int_{-2\pi W}^{2\pi W} d\lambda \, e^{i\lambda t} \sum_{m=-\infty}^{\infty} e^{i(\omega-\lambda)\frac{m}{2W}} \qquad .$$

It may be verified that

$$(2.3) \qquad \sum_{m=-\infty}^{\infty} e^{im\alpha} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\alpha + 2\pi n)$$

and, for any function $f(x)$,

$$(2.4) \qquad \int_I f(x) \, \delta(\frac{x-a}{c}) \, dx = c \, f(a)$$

if the point a lies in the interval of integration I; otherwise the integral in (2.4) is zero. Now, for $|\omega| < 2\pi W$, the only value of n such that $|\omega + 4\pi nW| \le 2\pi W$ is $n = 0$. Consequently, in view of (2.3) and (2.4),

it is seen that the value of (2.2) is $e^{i\omega t}$, for $|\omega| < 2\pi W$, which verifies (2.1).

Next, to prove Theorems I and II, merely replace $e^{i\omega t}$ in (1.1) and (1.5) by the infinite series of (2.1); that the order of integration and summation may be interchanged follows rigorously by the theory of Lebesgue integration.

Next, to prove Theorem III, we first note the fact that, from (1.7) it follows that $f(t)$ may be represented as a stochastic integral as follows:

$$(2.6) \qquad f(t) = \int_{-2\pi W}^{2\pi W} e^{i\omega t} \, dZ(\omega) \qquad .$$

By the theory of stochastic integrals, (1.8) follows from (2.6) if one shows that

$$\lim_{M \to \infty} \int_{-2\pi W}^{2\pi W} \left| e^{i\omega t} - S_M(\omega, t) \right|^2 G(\omega) \, d\omega = 0$$

which follows from the facts stated at the beginning of this section.

3. __Proof of Theorem IV.__

To prove (1.12) we begin by conjecturing that for ω such that $2\pi W_0 < |\omega| < 2\pi(W_0 + W)$, $e^{i\omega t}$ may be written in the form

$$(3.1) \quad e^{i\omega t} = \sum_{m=-\infty}^{\infty} e^{i\omega \frac{m}{W}} s_1\left(t - \frac{m}{W}\right) + \sum_{m=-\infty}^{\infty} e^{i\omega\left(\frac{m}{W} + k\right)} s_2\left(t - \frac{m}{W} - k\right)$$

for some constant k, and functions $s_1(t)$ and $s_2(t)$ which are of the

form, for $i = 1,2$,

$$(3.2) \qquad s_i(t) = \int_I e^{it\omega} S_1(\omega)\, d\omega$$

where the region of integration I consists of the intervals $(2\pi W_o, 2\pi(W_o + W))$ and $(-2\pi(W_o + W), -2\pi W_o)$. In view of (3.2), the right hand side of (3.1) may be written

$$(3.3) \qquad \int_I d\lambda\, S_1(\lambda)\, e^{i\lambda t} \sum_m e^{i(\omega-\lambda)\frac{m}{W}}$$

$$+ \int_I d\lambda S_2(\lambda)\, e^{i\lambda t}\, e^{ik(\omega-\lambda)} \sum_m e^{i(\omega-\lambda)\frac{m}{W}} \quad .$$

We next evaluate (3.3) by means of (2.3) and (2.4).

It is readily verified that, for ω in I, $\omega + 2\pi n W$ lies in I only for $n = 0$, and for one other value of n , which we denote by $N(\omega)$, whose values are given in Table A in terms of the integer r given by (1.10).

<u>Table A</u>

For ω in the interval	$N(\omega) =$	$\omega + 2\pi N(\omega)W$ lies in the interval
J_1 : $2\pi W_o < \omega < 2\pi(rW-W_o)$	$-r$	$-J_1$
$-J_1$: $-2\pi(rW-W_o) < \omega < -2\pi W_o$	r	J_1
J_2 : $2\pi(rW-W_o) < \omega < 2\pi(W_o + W)$	$-(r+1)$	$-J_2$
$-J_2$: $-2\pi(W_o + W) < \omega < -2\pi(rW-W_o)$	$r+1$	J_2

Then, for ω in I, the value of (3.3) is given by

$$(3.4) \qquad 2\pi W \ e^{i\omega t} \ [S_1(\omega) + S_2(\omega)] + 2\pi W \ e^{it(\omega + 2\pi N(\omega)W)}$$

$$[S_1(\omega + 2\pi N(\omega)W) + e^{-i2\pi N(\omega)Wk} S_2(\omega + 2\pi N(\omega)W)] \quad .$$

Now, let $S_1(\omega)$ and $S_2(\omega)$ be defined so that they vanish for ω outside I, and for $\omega = \pm 2\pi(rW\_W_0)$, and for other values of ω in I satisfy the relations

$$(3.5) \quad 2\pi W \ S_1(\omega) = 1 - 2\pi W \ S_2(\omega), \quad 2\pi W \ S_2(\omega) = [1 - e^{i2\pi N(\omega)Wk}]^{-1}$$

where we now choose k so that rWk and $(r+1)Wk$ are not equal to any of the integers $0, \pm 1, \pm 2, \ldots$. In view of Table A it may be verified that the following relations hold for ω in I, where an asterisk denotes a complex conjugate:

$$N(\omega) = - N(-\omega), \ S_1(\omega) = S_2(-\omega)$$

$$S_2(\omega + 2\pi N(\omega)W) = S_2^*(\omega) = S_2(-\omega),$$

$$S_1(\omega + 2\pi N(\omega)W) = S_1^*(\omega) = S_1(-\omega),$$

$$S_1(\omega + 2\pi N(\omega)W) + e^{-i2\pi N(\omega)Wk} S_2(\omega + 2\pi N(\omega)W)$$

$$= [S_1(\omega) + e^{i2\pi N(\omega)Wk} S_2(\omega)]^* = 0 ,$$

$$2\pi W(S_1(\omega) + S_2(\omega)) = 1 \quad .$$

In view of these relations, the value of (3.4) is $e^{i\omega t}$ for ω in I, <u>except</u> perhaps for $\omega = \pm 2\pi(rW-W_o)$, and (3.1) holds with $s_1(t)$ and $s_2(t)$ given by (3.2) and $S_1(\omega)$ and $S_2(\omega)$ given by (3.5).

We now write (3.1) in a more convenient form. Since $S_1(\omega) = S_2(-\omega)$, it follows that $s_1(t) = s_2(-t)$. Define $s(t) = s_1(t)$. Then, since $S_1^*(\omega) = S_1(-\omega)$,

$$(3.6) \qquad s(t) = \int_I e^{i\omega t} S_1(\omega) \, d\omega = 2 \int_{+2\pi W_o}^{2\pi(W_o + W)} \text{Real } [e^{i\omega t} S_1(\omega)] \, d\omega \quad .$$

Now

$$(3.7) \qquad 2 \text{ Real } [e^{i\omega t} S_1(\omega)] = \frac{1}{2\pi W} \left[\frac{\cos \omega t - \cos (\omega t + 2\pi N(\omega)Wk)}{1 - \cos 2\pi N(\omega)Wk} \right]$$

$$= \frac{1}{2\pi W} \left[\cos \omega t + \sin \omega t \cot \pi N(\omega)Wk \right] \quad .$$

Thus (1.12) has been established for the complex exponentials $e^{i\omega t}$ for $2\pi W_o < |\omega| < 2\pi(W_o + W)$, with the possible exception of $\omega = \pm 2\pi(rW-W_o)$. To establish (1.12) for any function $f(t)$ satisfying (1.9), merely replace $e^{i\omega t}$ in (1.9) by the infinite series representing it.

REFERENCES

[1]   C. E. Shannon, Proc. Inst. Radio Engrs.. 37 (1949), p. 10.

[2]   S. Goldman, Information Theory, New York, 1953.

[3]   A. Kohlenberg, J. Appl. Physics, 24 (1953), p. 1432.

[4]   P. M. Woodward, Probability and Information Theory, London, 1953.

[5]   J. L. Doob, Stochastic Processes, New York, 1953, p. 95 and p. 473.

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A Mathematical Restatement of Hypotheses in

"A SIMPLE PROOF AND SOME EXTENSIONS

OF THE SAMPLING THEOREM"[1]

by

Emanuel Parzen

This report was written in a terminology which would presumably be
most readable to communication engineers. For the reader interested
in mathematical precision, the following <u>restatements</u> of hypotheses
should be noted.

In Theorem I, for (1.1) read

$$(1.1) \qquad f(t) = \int_{-2\pi W}^{2\pi W} e^{i\omega t} \, dV(\omega)$$

where $V(\omega)$ is a function of bounded variation continuous at $\omega = \pm 2\pi W$.

In Theorem II, for (1.5) read

$$(1.5) \quad f(t_1,\ldots,t_K) = \int_{-2\pi W_1}^{2\pi W_1} \cdots \int_{-2\pi W_K}^{2\pi W_K} e^{i(\omega_1 t_1 + \ldots + \omega_K t_K)} \, dV(\omega_1,\ldots,\omega_K)$$

where $V(\omega_1,\ldots,\omega_K)$ is a function of bounded variation which assigns
measure zero to the set $\{(\omega_1,\ldots,\omega_K) : \omega_i = \pm 2\pi W_i \text{ for some } i\}$.

In Theorem III, for (1.7) read

$$(1.7) \qquad R(\tau) = \int_{-2\pi W}^{2\pi W} e^{i\omega\tau} \, dF(\omega)$$

where the spectral distribution function $F(\omega)$ is monotone non-decreasing and continuous at $\omega = \pm 2\pi W$.

In Theorem IV, for (1.9) read

$$(1.9) \quad f(t) = \int_{2\pi W_0}^{2\pi(W_0 + W)} e^{i\omega t} \, dV(\omega) + \int_{-2\pi(W_0 + W)}^{-2\pi W_0} e^{i\omega t} \, dV(\omega)$$

where $V(\omega)$ is a function of bounded variation continuous at $\omega = \pm 2\pi W_0$, $\omega = \pm 2\pi(W_0 + W)$, and $\omega = \pm 2\pi(rW - W_0)$, where r is defined by (1.10).